A remark on the negation in bilateral state-based modal logic

Aleksi Anttila
University of Helsinki

NihiL seminar
ILLC, University of Amsterdam

Overview

The **dual negation** $\neg$ is the negation used in the original formulations of first-order dependence logic $\mathcal{D}$. This kind of notion of negation is naturally induced by game-theoretic semantics.

In the classical fragment of $\mathcal{D}$, the dual negation is equivalent to the classical negation.

Overview

The **dual negation** $\neg$ is the negation used in the original formulations of first-order dependence logic $\mathcal{D}$. This kind of notion of negation is naturally induced by game-theoretic semantics.

In the classical fragment of $\mathcal{D}$, the dual negation is equivalent to the classical negation.

But for arbitrary sentences ϕ and ψ, $\phi \equiv \psi$ does not imply $\neg\phi \equiv \neg\psi$. In other words, the class of models $||\phi||$ of ϕ does not determine $||\neg\phi||$. So $\neg$ does not correspond to any well-defined semantic operation, whereas e.g. $||\phi \wedge \psi|| = ||\phi|| \cap ||\psi||$.

Overview

The **dual negation** $\neg$ is the negation used in the original formulations of first-order dependence logic $\mathcal{D}$. This kind of notion of negation is naturally induced by game-theoretic semantics.

In the classical fragment of $\mathcal{D}$, the dual negation is equivalent to the classical negation.

But for arbitrary sentences ϕ and ψ, $\phi \equiv \psi$ does not imply $\neg\phi \equiv \neg\psi$. In other words, the class of models $||\phi||$ of ϕ does not determine $||\neg\phi||$. So $\neg$ does not correspond to any well-defined semantic operation, whereas e.g. $||\phi \wedge \psi|| = ||\phi|| \cap ||\psi||$.

Burgess (2003) showed (in the equivalent context of Henkin sentences) that this lack of determination is extreme: for any sentences ϕ and ψ that share no models, there is some sentence θ such that $\theta \equiv \phi$ and $\neg\theta \equiv \psi$. So given only $||\phi||$, we do not know anything about $||\neg\phi||$ except $||\phi|| \cap ||\neg\phi|| = \varnothing$ (and that $||\neg\phi||$ is expressible in $\mathcal{D}$). Kontinen & Väänänen (2011) generalized this to open formulas.

Overview

The **dual negation** $\neg$ is the negation used in the original formulations of first-order dependence logic $\mathcal{D}$. This kind of notion of negation is naturally induced by game-theoretic semantics.

In the classical fragment of $\mathcal{D}$, the dual negation is equivalent to the classical negation.

But for arbitrary sentences ϕ and ψ, $\phi \equiv \psi$ does not imply $\neg\phi \equiv \neg\psi$. In other words, the class of models $||\phi||$ of ϕ does not determine $||\neg\phi||$. So $\neg$ does not correspond to any well-defined semantic operation, whereas e.g. $||\phi \wedge \psi|| = ||\phi|| \cap ||\psi||$.

Burgess (2003) showed (in the equivalent context of Henkin sentences) that this lack of determination is extreme: for any sentences ϕ and ψ that share no models, there is some sentence θ such that $\theta \equiv \phi$ and $\neg\theta \equiv \psi$. So given only $||\phi||$, we do not know anything about $||\neg\phi||$ except $||\phi|| \cap ||\neg\phi|| = \varnothing$ (and that $||\neg\phi||$ is expressible in $\mathcal{D}$). Kontinen & Väänänen (2011) generalized this to open formulas.

Aloni's (2022) Bilateral State-based Modal Logic (**BSML**) makes use of a bilateral negation which is essentially the same notion as the dual negation. **BSML** differs from $\mathcal{D}$ in being modal rather than first-order, and not being downward closed. We show that Burgess' result holds for **BSML** and an extension of **BSML**.

Syntax of **first-order dependence logic** $\mathcal{D}$ without the dual negation:

$$\phi \quad := \quad t_1 = t_2 \mid \neg(t_1 = t_2) \mid R\vec{t} \mid \neg R\vec{t} \mid \; =(t_1, \ldots, t_n, t) \mid \phi \wedge \psi \mid \phi \vee \psi \mid \exists x\phi \mid \forall x\phi$$

Where the t_i are *FO* terms. I.e. we have FO formulas together with **dependence atoms** $=(t_1, \ldots, t_n, t)$; negation is only allowed to occur in front of atomic FO formulas.

Syntax of **first-order dependence logic** $\mathcal{D}$ without the dual negation:

$$\phi \quad := \quad t_1 = t_2 \mid \neg(t_1 = t_2) \mid R\vec{t} \mid \neg R\vec{t} \mid\ =\!(t_1, \ldots, t_n, t) \mid \phi \wedge \psi \mid \phi \vee \psi \mid \exists x \phi \mid \forall x \phi$$

Where the t_i are *FO* terms. I.e. we have FO formulas together with **dependence atoms** $=\!(t_1, \ldots, t_n, t)$; negation is only allowed to occur in front of atomic FO formulas.

Intuitive meaning of $=\!(t_1, \ldots, t_n, t)$: the value of t is completely determined by the values of $t_1, \ldots, t_n$.

Syntax of **first-order dependence logic** $\mathcal{D}$ without the dual negation:

$$\phi \quad := \quad t_1 = t_2 \mid \neg(t_1 = t_2) \mid R\vec{t} \mid \neg R\vec{t} \mid \; =\!(t_1, \ldots, t_n, t) \mid \phi \wedge \psi \mid \phi \vee \psi \mid \exists x \phi \mid \forall x \phi$$

Where the t_i are *FO* terms. I.e. we have FO formulas together with **dependence atoms** $=\!(t_1, \ldots, t_n, t)$; negation is only allowed to occur in front of atomic FO formulas.

Intuitive meaning of $=\!(t_1, \ldots, t_n, t)$: the value of t is completely determined by the values of $t_1, \ldots, t_n$.

Team semantics: formulas are interpreted with respect to teams. Given a model $\mathcal{M}$ and set of variables V, a **team** X of $\mathcal{M}$ with domain V is a set of assignments $s : V \to dom(\mathcal{M})$. The interpretation $s(t^{\mathcal{M}})$ of t under $\mathcal{M}$ and s is defined as usual.

Syntax of **first-order dependence logic** $\mathcal{D}$ without the dual negation:

$$\phi \quad ::= \quad t_1 = t_2 \mid \neg(t_1 = t_2) \mid R\vec{t} \mid \neg R\vec{t} \mid \ =\!(t_1, \ldots, t_n, t) \mid \phi \wedge \psi \mid \phi \vee \psi \mid \exists x\phi \mid \forall x\phi$$

Where the t_i are *FO* terms. I.e. we have FO formulas together with **dependence atoms** $=\!(t_1, \ldots, t_n, t)$; negation is only allowed to occur in front of atomic FO formulas.

Intuitive meaning of $=\!(t_1, \ldots, t_n, t)$: the value of t is completely determined by the values of $t_1, \ldots, t_n$.

Team semantics: formulas are interpreted with respect to teams. Given a model $\mathcal{M}$ and set of variables V, a **team** X of $\mathcal{M}$ with domain V is a set of assignments $s : V \to dom(\mathcal{M})$. The interpretation $s(t^{\mathcal{M}})$ of t under $\mathcal{M}$ and s is defined as usual.

$\mathcal{M} \models_X =\!(t_1, \ldots, t_n, t)$ iff $\forall s, s' \in X$: if $s(t_1^{\mathcal{M}}) = s'(t_1^{\mathcal{M}}), \ldots, s(t_n^{\mathcal{M}}) = s'(t_n^{\mathcal{M}})$, then $s(t^{\mathcal{M}}) = s'(t^{\mathcal{M}})$.

	x	y	z
s_1	a	b	b
s_2	a	b	c

In the team $X = \{s_1, s_2\}$, $X \models =\!(x, y)$ and $X \not\models =\!(x, z)$. $X \models =\!(y)$ because the value of y is constant in X.

Given a model $\mathcal{M}$ with domain M, a team X of $\mathcal{M}$ and $F : X \to M$ let:

$$X(F/x) := \{s(F(s)/x) \mid s \in X\}$$
$$X(M/x) := \{s(a/x) \mid a \in M, s \in X\}$$

Given a model $\mathcal{M}$ with domain M, a team X of $\mathcal{M}$ and $F : X \to M$ let:

$$X(F/x) := \{s(F(s)/x) \mid s \in X\}$$
$$X(M/x) := \{s(a/x) \mid a \in M, s \in X\}$$

Team X of $\mathcal{M}$ where $M = \{a, b\}$

	x
s_1	b
s_2	a

$X(F/y)(M/z)$ where $F(s_1) = a$, $F(s_2) = b$

	x	y	z
s_1'	b	a	a
s_2'	b	a	b
s_3'	a	b	a
s_4'	a	b	b

Given a model $\mathcal{M}$ with domain M, a team X of $\mathcal{M}$ and $F : X \to M$ let:

$$X(F/x) := \{s(F(s)/x) \mid s \in X\}$$
$$X(M/x) := \{s(a/x) \mid a \in M, s \in X\}$$

Team X of $\mathcal{M}$ where $M = \{a, b\}$

	x
s_1	b
s_2	a

$X(F/y)(M/z)$ where $F(s_1) = a$, $F(s_2) = b$

	x	y	z
s_1'	b	a	a
s_2'	b	a	b
s_3'	a	b	a
s_4'	a	b	b

We define $\mathcal{M} \vDash_X \phi$ by:

$\mathcal{M} \vDash_X \alpha$ iff $\forall s \in X : \mathcal{M} \vDash_s \alpha$ for α an FO atom or negated FO atom

$\mathcal{M} \vDash_X {=}(t_1, \ldots, t_n, t)$ iff $\forall s, s' \in X$: if $s(t_1^{\mathcal{M}}) = s'(t_1^{\mathcal{M}}) \ldots s(t_n^{\mathcal{M}}) = s'(t_n^{\mathcal{M}})$ then $s(t^{\mathcal{M}}) = s'(t^{\mathcal{M}})$

$\mathcal{M} \vDash_X \phi \wedge \psi$ iff $\mathcal{M} \vDash_X \phi$ and $\mathcal{M} \vDash_X \psi$

$\mathcal{M} \vDash_X \phi \vee \psi$ iff $\exists Y, Z : X = Y \cup Z$ and $\mathcal{M} \vDash_Y \phi$ and $\mathcal{M} \vDash_Z \psi$

$\mathcal{M} \vDash_X \exists x\phi$ iff $\mathcal{M} \vDash_{X(F/x)} \phi$ for some $F : X \to M$

$\mathcal{M} \vDash_X \forall x\phi$ iff $\mathcal{M} \vDash_{X(M/x)} \phi$

A sentence ϕ is **true** in $\mathcal{M}$ ($\mathcal{M} \vDash \phi$) iff $\mathcal{M} \vDash_{\{\varnothing\}} \phi$. $\{\varnothing\}$ contains only the empty assignment.

To get $\mathcal{D}$ with the dual negation, allow $\neg$ to appear anywhere and define both a positive semantic notion $\vDash_X$ and a negative notion $\Dashv_X$:

$$
\begin{array}{lll}
\mathcal{M} \vDash_X \alpha & \text{iff} & \forall s \in X : \mathcal{M} \vDash_s \alpha \quad \text{for } \alpha \text{ an FO atom or negated FO atom} \\
\mathcal{M} \Dashv_X \alpha & \text{iff} & \forall s \in X : \mathcal{M} \nvDash_s \alpha \quad \text{for } \alpha \text{ an FO atom or negated FO atom} \\
\mathcal{M} \vDash_X =\!(t_1, \dots, t_n, t) & \text{iff} & \forall s, s' \in X : \text{ if } s(t_1^{\mathcal{M}}) = s'(t_1^{\mathcal{M}}) \dots s(t_n^{\mathcal{M}}) = s'(t_n^{\mathcal{M}}) \text{ then } s(t^{\mathcal{M}}) = s'(t^{\mathcal{M}}) \\
\mathcal{M} \Dashv_X =\!(t_1, \dots, t_n, t) & \text{iff} & X = \varnothing \\
\mathcal{M} \vDash_X \phi \vee \psi & \text{iff} & \exists Y, Z : X = Y \cup Z \text{ and } \mathcal{M} \vDash_Y \psi \text{ and } \mathcal{M} \vDash_Z \psi \\
\mathcal{M} \Dashv_X \phi \vee \psi & \text{iff} & \mathcal{M} \Dashv_X \phi \text{ and } \mathcal{M} \Dashv_X \psi \\
\mathcal{M} \vDash_X \exists x \phi & \text{iff} & \mathcal{M} \vDash_{X(F/x)} \phi \text{ for some } F : X \to M \\
\mathcal{M} \Dashv_X \exists x \phi & \text{iff} & \mathcal{M} \Dashv_{X(M/x)} \phi \\
\mathcal{M} \vDash_X \neg \phi & \text{iff} & \mathcal{M} \Dashv_X \phi \\
\mathcal{M} \Dashv_X \neg \phi & \text{iff} & \mathcal{M} \vDash_X \phi
\end{array}
$$

(We can define $\wedge := \neg \vee \neg$ and $\forall := \neg \exists \neg$.)

The dual negation arises naturally in the context of **game-theoretic semantics for $\mathcal{D}$**: "the game-theoretic intuition behind $\neg\phi$ is that it says something about the other player." (Väänänen 2007)

The dual negation arises naturally in the context of **game-theoretic semantics for $\mathcal{D}$**: "the game-theoretic intuition behind $\neg\phi$ is that it says something about the other player." (Väänänen 2007)

A semantic game for $\mathcal{D}$ has two players, I and II. For a given $\mathcal{M}$, a position in the game $G(\phi)$ is a triple (ψ, X, d) where ψ is a formula, X a team on $\mathcal{M}$ and $d \in \{0,1\}$. $G(\phi)$ is defined as follows. The starting position is $(\phi, \{\varnothing\}, 1\}$. Given position (ψ, X, d):

If ψ is a FO atom and $d = 1$, the game ends. II wins if $\forall s \in X : \mathcal{M} \vDash_s \psi$; otherwise I wins.

If ψ is a FO atom and $d = 0$, the game ends. II wins if $\forall s \in X : \mathcal{M} \nvDash_s \psi$; otherwise I wins.

If ψ is $=\!(t_1, \dots, t_n, t)$ and $d = 1$, the game ends. II if $\mathcal{M} \vDash_X =\!(t_1, \dots, t_n, t)$; otherwise I wins.

If ψ is $=\!(t_1, \dots, t_n, t)$ and $d = 0$, the game ends. II if $X = \varnothing$; otherwise I wins.

If $\psi = \chi \vee \eta$ and $d = 1$, II chooses Y, Z s.t. $X = Y \cup Z$. I chooses whether the game continues from $(\chi, Y, 1)$ or $(\eta, Y, 1)$.

If $\psi = \chi \vee \eta$ and $d = 0$, I chooses whether the game continues from $(\chi, X, 0)$ or $(\eta, X, 0)$.

If $\psi = \exists x \chi$ and $d = 1$, II chooses $F : X \to M$ and the game continues from $(\chi, X(F/x), 1)$.

If $\psi = \exists x \chi$ and $d = 0$, the game continues from $(\chi, X(M/x), 0)$.

If $\psi = \neg\chi$ and $d = 1$, the game continues from $(\chi, X, 0)$.

If $\psi = \neg\chi$ and $d = 0$, the game continues from $(\chi, X, 1)$.

Let $\phi \models \psi$ iff $\forall \mathcal{M} : \forall X$ on $\mathcal{M}$: $\mathcal{M} \models_X \phi$ implies $\mathcal{M} \models_X \psi$; and $\phi \equiv \psi$ iff $\phi \models \psi$ and $\psi \models \phi$.
We have the following equivalences:

$$
\begin{aligned}
\neg\neg\phi &\equiv \phi \\
\neg(\phi \vee \psi) &\equiv \neg\phi \wedge \neg\psi \\
\neg(\phi \wedge \psi) &\equiv \neg\phi \vee \neg\psi \\
\neg\exists x\phi &\equiv \forall x\neg\phi \\
\neg\forall x\phi &\equiv \exists x\neg\phi
\end{aligned}
$$

Let $\phi \vDash \psi$ iff $\forall \mathcal{M} : \forall X$ on $\mathcal{M}$: $\mathcal{M} \vDash_X \phi$ implies $\mathcal{M} \vDash_X \psi$; and $\phi \equiv \psi$ iff $\phi \vDash \psi$ and $\psi \vDash \phi$. We have the following equivalences:

$$
\begin{array}{lcl}
\neg\neg\phi & \equiv & \phi \\
\neg(\phi \vee \psi) & \equiv & \neg\phi \wedge \neg\psi \\
\neg(\phi \wedge \psi) & \equiv & \neg\phi \vee \neg\psi \\
\neg\exists x\phi & \equiv & \forall x\neg\phi \\
\neg\forall x\phi & \equiv & \exists x\neg\phi
\end{array}
$$

So a simpler, equivalent way of defining the dual negation is as follows. Only define $\mathcal{M} \vDash_X \neg\phi$ when ϕ is an atom:

$$
\begin{array}{llll}
\mathcal{M} \vDash_X \neg\alpha & \text{iff} & \forall s \in X : \mathcal{M} \nvDash_s \alpha & \text{for } \alpha \text{ an FO atom} \\
\mathcal{M} \vDash_X \neg =\!(t_1, \ldots, t_n, t) & \text{iff} & X = \varnothing &
\end{array}
$$

and for other negated formulas $\neg\phi$, take $\neg\phi$ to be an abbreviation of a formula in negation normal form acquired by employing the equivalences above.

Examples:

	x
s_1	b
s_2	a

Here $\mathcal{M} \not\models_X (x = a) \wedge {=}(x)$ and also $\mathcal{M} \not\models_X \neg((x = a) \wedge {=}(x))$:

$$\mathcal{M} \models_X \neg((x = a) \wedge {=}(x)) \iff \mathcal{M} \models_X \neg(x = a) \vee \neg {=}(x)$$
$$\iff \exists Y, Z : X = Y \cup Z \text{ and } \mathcal{M} \models_Y \neg(x = a) \text{ and } Z = \varnothing$$

Examples:

	x
s_1	b
s_2	a

Here $\mathcal{M} \not\models_X (x = a) \wedge {=}(x)$ and also $\mathcal{M} \not\models_X \neg((x = a) \wedge {=}(x))$:

$$\mathcal{M} \models_X \neg((x = a) \wedge {=}(x)) \iff \mathcal{M} \models_X \neg(x = a) \vee \neg {=}(x)$$
$$\iff \exists Y, Z : X = Y \cup Z \text{ and } \mathcal{M} \models_Y \neg(x = a) \text{ and } Z = \varnothing$$

Let $\mathcal{M}$ be a model with $|M| \geq 2$. Let $\theta_0 := \forall x \, {=}(x)$. Then:

$$\begin{aligned}
\mathcal{M} \models_X \theta_0 \quad & \iff \mathcal{M} \models_X \forall x \, {=}(x) \iff \mathcal{M} \models_{X(M/x)} {=}(x) \\
& \iff \forall s \in X : \forall a, b \in M : s(a/x) = s(b/x) \qquad \iff X = \varnothing \\
\mathcal{M} \models_X \neg\theta_0 \quad & \iff \mathcal{M} \models_X \neg\forall x \, {=}(x) \iff \mathcal{M} \models_X \exists x \neg {=}(x) \\
& \iff \exists F : X \to M : \mathcal{M} \dashv_{X(F/x)} {=}(x) \\
& \iff \exists F : X \to M : X(F/x) = \varnothing \qquad \iff X = \varnothing
\end{aligned}$$

Some properties and results:

The empty dependence atom $=()$ is always true. Denote $\bot := \neg =()$. Then $\bot \equiv \neg =(x)$ but $=() \equiv \neg\bot \not\equiv \neg\neg =(x) \equiv =(x)$. So $\phi \equiv \psi \not\Longrightarrow \neg\phi \equiv \neg\psi$.

Some properties and results:

The empty dependence atom $=()$ is always true. Denote $\bot := \neg =()$. Then $\bot \equiv \neg =(x)$ but $=() \equiv \neg\bot \not\equiv \neg\neg =(x) \equiv =(x)$. So $\phi \equiv \psi \not\Longrightarrow \neg\phi \equiv \neg\psi$.

On the other hand, let ϕ and ψ be **strongly equivalent** $\phi \equiv^* \psi$ iff $\phi \equiv \psi$ and $\neg\phi \equiv \neg\psi$. Then $\phi \equiv^* \psi \implies \neg\phi \equiv^* \neg\psi$ and more generally $\phi(\vec{x}) \equiv^* \psi(\vec{x}) \implies \chi[\phi(\vec{x})/P\vec{x}] \equiv^* \chi[\psi(\vec{x})/P\vec{x}]$.

Some properties and results:

The empty dependence atom $=()$ is always true. Denote $\bot := \neg =()$. Then $\bot \equiv \neg =(x)$ but $=() \equiv \neg\bot \not\equiv \neg\neg =(x) \equiv =(x)$. So $\phi \equiv \psi \not\Longrightarrow \neg\phi \equiv \neg\psi$.

On the other hand, let ϕ and ψ be **strongly equivalent** $\phi \equiv^* \psi$ iff $\phi \equiv \psi$ and $\neg\phi \equiv \neg\psi$. Then $\phi \equiv^* \psi \implies \neg\phi \equiv^* \neg\psi$ and more generally $\phi(\vec{x}) \equiv^* \psi(\vec{x}) \implies \chi[\phi(\vec{x})/P\vec{x}] \equiv^* \chi[\psi(\vec{x})/P\vec{x}]$.

α is **first order/classical** if no dependence atoms appear in α. Classical formulas α are **flat**: $\mathcal{M} \vDash_X \alpha \iff \forall s \in X : \mathcal{M} \vDash_s \alpha$. In particular, the dual negation coincides with the classical negation for classical formulas: $\mathcal{M} \vDash_X \neg\alpha \iff \forall s \in X : \mathcal{M} \vDash_s \neg\alpha$.

Some properties and results:

The empty dependence atom $=()$ is always true. Denote $\bot := \neg =()$. Then $\bot \equiv \neg =(x)$ but $=() \equiv \neg\bot \not\equiv \neg\neg =(x) \equiv =(x)$. So $\phi \equiv \psi \not\Longrightarrow \neg\phi \equiv \neg\psi$.

On the other hand, let ϕ and ψ be **strongly equivalent** $\phi \equiv^* \psi$ iff $\phi \equiv \psi$ and $\neg\phi \equiv \neg\psi$. Then $\phi \equiv^* \psi \implies \neg\phi \equiv^* \neg\psi$ and more generally
$\phi(\vec{x}) \equiv^* \psi(\vec{x}) \implies \chi[\phi(\vec{x})/P\vec{x}] \equiv^* \chi[\psi(\vec{x})/P\vec{x}]$.

α is **first order/classical** if no dependence atoms appear in α. Classical formulas α are **flat**: $\mathcal{M} \vDash_X \alpha \iff \forall s \in X : \mathcal{M} \vDash_s \alpha$. In particular, the dual negation coincides with the classical negation for classical formulas: $\mathcal{M} \vDash_X \neg\alpha \iff \forall s \in X : \mathcal{M} \vDash_s \neg\alpha$.

Empty team property: $\mathcal{M} \vDash_\varnothing \phi$ for all ϕ. (Note that the team $\varnothing$ is not the team $\{\varnothing\}$).

Some properties and results:

The empty dependence atom $=()$ is always true. Denote $\bot := \neg =()$. Then $\bot \equiv \neg =(x)$ but $=() \equiv \neg\bot \not\equiv \neg\neg =(x) \equiv =(x)$. So $\phi \equiv \psi \not\Longrightarrow \neg\phi \equiv \neg\psi$.

On the other hand, let ϕ and ψ be **strongly equivalent** $\phi \equiv^* \psi$ iff $\phi \equiv \psi$ and $\neg\phi \equiv \neg\psi$. Then $\phi \equiv^* \psi \implies \neg\phi \equiv^* \neg\psi$ and more generally $\phi(\vec{x}) \equiv^* \psi(\vec{x}) \implies \chi[\phi(\vec{x})/P\vec{x}] \equiv^* \chi[\psi(\vec{x})/P\vec{x}]$.

α is **first order/classical** if no dependence atoms appear in α. Classical formulas α are **flat**: $\mathcal{M} \vDash_X \alpha \iff \forall s \in X : \mathcal{M} \vDash_s \alpha$. In particular, the dual negation coincides with the classical negation for classical formulas: $\mathcal{M} \vDash_X \neg\alpha \iff \forall s \in X : \mathcal{M} \vDash_s \neg\alpha$.

Empty team property: $\mathcal{M} \vDash_\varnothing \phi$ for all ϕ. (Note that the team $\varnothing$ is not the team $\{\varnothing\}$).

Downward closure: $\mathcal{M} \vDash_X \phi$ and $Y \subseteq X$ implies $\mathcal{M} \vDash_Y \phi$.

Some properties and results:

The empty dependence atom $=()$ is always true. Denote $\bot := \neg =()$. Then $\bot \equiv \neg =(x)$ but $=() \equiv \neg\bot \not\equiv \neg\neg =(x) \equiv =(x)$. So $\phi \equiv \psi \not\Longrightarrow \neg\phi \equiv \neg\psi$.

On the other hand, let ϕ and ψ be **strongly equivalent** $\phi \equiv^* \psi$ iff $\phi \equiv \psi$ and $\neg\phi \equiv \neg\psi$. Then $\phi \equiv^* \psi \implies \neg\phi \equiv^* \neg\psi$ and more generally $\phi(\vec{x}) \equiv^* \psi(\vec{x}) \implies \chi[\phi(\vec{x})/P\vec{x}] \equiv^* \chi[\psi(\vec{x})/P\vec{x}]$.

α is **first order/classical** if no dependence atoms appear in α. Classical formulas α are **flat**: $\mathcal{M} \vDash_X \alpha \iff \forall s \in X : \mathcal{M} \vDash_s \alpha$. In particular, the dual negation coincides with the classical negation for classical formulas: $\mathcal{M} \vDash_X \neg\alpha \iff \forall s \in X : \mathcal{M} \vDash_s \neg\alpha$.

Empty team property: $\mathcal{M} \vDash_\varnothing \phi$ for all ϕ. (Note that the team $\varnothing$ is not the team $\{\varnothing\}$).

Downward closure: $\mathcal{M} \vDash_X \phi$ and $Y \subseteq X$ implies $\mathcal{M} \vDash_Y \phi$.

Expressive equivalence with Σ_1^1 over sentences (applies both to $\mathcal{D}$ both with and $\mathcal{D}$ without the dual negation):

For any $\phi \in \mathcal{D}$ there is a $\phi_\chi \in \Sigma_1^1$ (in the same vocabulary) s.t. $\mathcal{M} \vDash \phi \iff \mathcal{M} \vDash \phi_\chi$.

For any $\phi \in \Sigma_1^1$ there is a $\chi_\phi \in \mathcal{D}$ (in the same vocabulary) s.t. $\mathcal{M} \vDash \phi \iff \mathcal{M} \vDash \chi_\phi$.

Burgess' result: Let ϕ, ψ be sentences of $\mathcal{D}$. The following are equivalent:

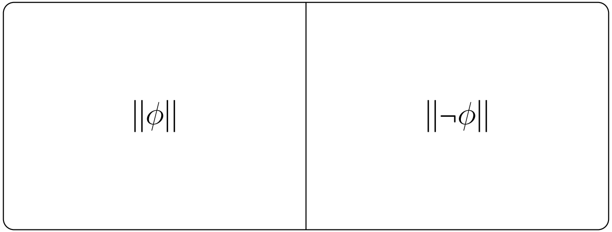
1. ϕ and ψ are contradictory in that $\phi, \psi \vDash \bot$ (i.e. $\mathcal{M} \vDash \phi$ iff $\mathcal{M} \nvDash \psi$).
2. There is a sentence $\theta \in \mathcal{D}$ such that $\phi \equiv \theta$ and $\psi \equiv \neg\theta$.

Suppose we know the set $\|\phi\| = \{\mathcal{M} \mid \mathcal{M} \vDash \phi\}$ of models on which a sentence ϕ is true (without knowing ϕ) and we want to work out $\|\neg\phi\|$.

Burgess' result: Let ϕ, ψ be sentences of $\mathcal{D}$. The following are equivalent:

1. ϕ and ψ are contradictory in that $\phi, \psi \vDash \bot$ (i.e. $\mathcal{M} \vDash \phi$ iff $\mathcal{M} \nvDash \psi$).
2. There is a sentence $\theta \in \mathcal{D}$ such that $\phi \equiv \theta$ and $\psi \equiv \neg\theta$.

Suppose we know the set $||\phi|| = \{\mathcal{M} \mid \mathcal{M} \vDash \phi\}$ of models on which a sentence ϕ is true (without knowing ϕ) and we want to work out $||\neg\phi||$. If ϕ is classical, we know $\mathcal{M} \vDash \neg\phi \iff \mathcal{M} \nvDash \phi$.

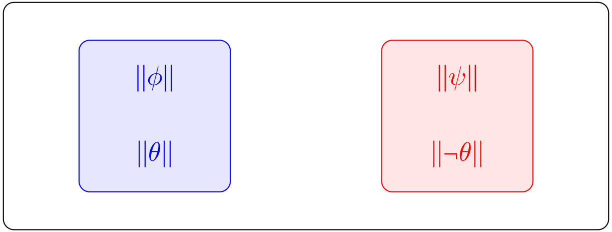


Burgess' result: Let ϕ, ψ be sentences of $\mathcal{D}$. The following are equivalent:

1. ϕ and ψ are contradictory in that $\phi, \psi \vDash \bot$ (i.e. $\mathcal{M} \vDash \phi$ iff $\mathcal{M} \nvDash \psi$).
2. There is a sentence $\theta \in \mathcal{D}$ such that $\phi \equiv \theta$ and $\psi \equiv \neg\theta$.

Suppose we know the set $||\phi|| = \{\mathcal{M} \mid \mathcal{M} \vDash \phi\}$ of models on which a sentence ϕ is true (without knowing ϕ) and we want to work out $||\neg\phi||$. If ϕ is classical, we know $\mathcal{M} \vDash \neg\phi \iff \mathcal{M} \nvDash \phi$.

Burgess: for any ϕ and ψ, if $||\phi||$ and $||\psi||$ are disjoint, there is θ with $||\theta|| = ||\phi||$ and $||\neg\theta|| = ||\psi||$.

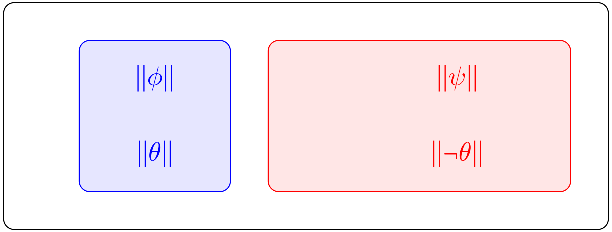


Burgess' result: Let ϕ, ψ be sentences of $\mathcal{D}$. The following are equivalent:

1. ϕ and ψ are contradictory in that $\phi, \psi \vDash \bot$ (i.e. $\mathcal{M} \vDash \phi$ iff $\mathcal{M} \nvDash \psi$).
2. There is a sentence $\theta \in \mathcal{D}$ such that $\phi \equiv \theta$ and $\psi \equiv \neg\theta$.

Suppose we know the set $||\phi|| = \{\mathcal{M} \mid \mathcal{M} \vDash \phi\}$ of models on which a sentence ϕ is true (without knowing ϕ) and we want to work out $||\neg\phi||$. If ϕ is classical, we know $\mathcal{M} \vDash \neg\phi \iff \mathcal{M} \nvDash \phi$.

Burgess: for any ϕ and ψ, if $||\phi||$ and $||\psi||$ are disjoint, there is θ with $||\theta|| = ||\phi||$ and $||\neg\theta|| = ||\psi||$.

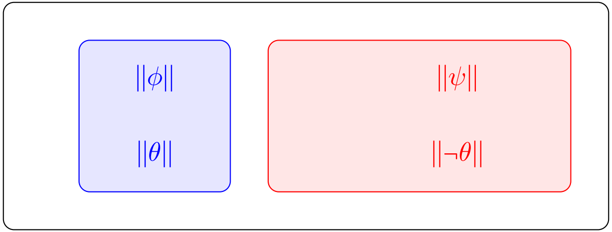


Burgess' result: Let ϕ, ψ be sentences of $\mathcal{D}$. The following are equivalent:

1. ϕ and ψ are contradictory in that $\phi, \psi \vDash \bot$ (i.e. $\mathcal{M} \vDash \phi$ iff $\mathcal{M} \nvDash \psi$).
2. There is a sentence $\theta \in \mathcal{D}$ such that $\phi \equiv \theta$ and $\psi \equiv \neg\theta$.

Suppose we know the set $||\phi|| = \{\mathcal{M} \mid \mathcal{M} \vDash \phi\}$ of models on which a sentence ϕ is true (without knowing ϕ) and we want to work out $||\neg\phi||$. If ϕ is classical, we know $\mathcal{M} \vDash \neg\phi \iff \mathcal{M} \nvDash \phi$.

Burgess: for any ϕ and ψ, if $||\phi||$ and $||\psi||$ are disjoint, there is θ with $||\theta|| = ||\phi||$ and $||\neg\theta|| = ||\psi||$.



So given only $||\phi||$, $||\neg\phi||$ can be any set of models X, as long as that set is definable in $\mathcal{D}$ ($X = ||\psi||$) and $||\phi|| \cap X = \varnothing$.

Separation theorem: Let ϕ, ψ be sentences of $\mathcal{D}$ with τ the vocabulary of ϕ and τ' the vocabulary of ψ. If ϕ and ψ are contradictory in that $\phi, \psi \models \bot$ (i.e. $\mathcal{M} \models \phi$ iff $\mathcal{M} \not\models \psi$), then there is a first-order sentence η in the vocabulary $\tau \cap \tau'$ such that $\phi \models \eta$ and $\psi \models \neg\eta$.

Separation theorem: Let ϕ, ψ be sentences of $\mathcal{D}$ with τ the vocabulary of ϕ and τ' the vocabulary of ψ. If ϕ and ψ are contradictory in that $\phi, \psi \vDash \bot$ (i.e. $\mathcal{M} \vDash \phi$ iff $\mathcal{M} \nvDash \psi$), then there is a first-order sentence η in the vocabulary $\tau \cap \tau'$ such that $\phi \vDash \eta$ and $\psi \vDash \neg\eta$.

Proof.

By expressive equivalence with Σ_1^1, there are $\exists \vec{S}\alpha, \exists \vec{T}\beta \in \Sigma_1^1$ such that $\phi \equiv \exists \vec{S}\alpha$ and α is FO in $\tau \cup \{S_1, \dots S_n\}$; and $\psi \equiv \exists \vec{T}\beta$ and β is FO in $\tau' \cup \{T_1, \dots T_m\}$. We can assume the sets $\{S_1, \dots S_n\}$ and $\{T_1, \dots T_m\}$ are disjoint.

Separation theorem: Let ϕ, ψ be sentences of $\mathcal{D}$ with τ the vocabulary of ϕ and τ' the vocabulary of ψ. If ϕ and ψ are contradictory in that $\phi, \psi \vDash \bot$ (i.e. $\mathcal{M} \vDash \phi$ iff $\mathcal{M} \nvDash \psi$), then there is a first-order sentence η in the vocabulary $\tau \cap \tau'$ such that $\phi \vDash \eta$ and $\psi \vDash \neg\eta$.

Proof.

By expressive equivalence with Σ_1^1, there are $\exists\vec{S}\alpha, \exists\vec{T}\beta \in \Sigma_1^1$ such that $\phi \equiv \exists\vec{S}\alpha$ and α is FO in $\tau \cup \{S_1, \ldots S_n\}$; and $\psi \equiv \exists\vec{T}\beta$ and β is FO in $\tau' \cup \{T_1, \ldots T_m\}$. We can assume the sets $\{S_1, \ldots S_n\}$ and $\{T_1, \ldots T_m\}$ are disjoint.

Since $\phi \equiv \exists\vec{S}\alpha$ and $\psi \equiv \exists\vec{T}\beta$, we have $\alpha \vDash \neg\beta$. By Craig's interpolation for FO, there is a FO sentence η in $(\tau \cup \{S_1, \ldots, S_n\}) \cap (\tau' \cup \{T_1, \ldots, T_n\}) = \tau \cap \tau'$ such that $\alpha \vDash \eta$ and $\eta \vDash \neg\beta$. Then also $\phi \vDash \eta$ and $\psi \vDash \neg\eta$. $\square$

Burgess' result: Let ϕ, ψ be sentences of $\mathcal{D}$. The following are equivalent:

1. ϕ and ψ are contradictory in that $\phi, \psi \vDash \bot$ (i.e. $\mathcal{M} \vDash \phi$ iff $\mathcal{M} \nvDash \psi$).
2. There is a sentence $\theta \in \mathcal{D}$ such that $\phi \equiv \theta$ and $\psi \equiv \neg\theta$.

We assume $|M| \geq 2$ for all models $\mathcal{M}$.

Burgess' result: Let ϕ, ψ be sentences of $\mathcal{D}$. The following are equivalent:

1. ϕ and ψ are contradictory in that $\phi, \psi \models \bot$ (i.e. $\mathcal{M} \models \phi$ iff $\mathcal{M} \not\models \psi$).
2. There is a sentence $\theta \in \mathcal{D}$ such that $\phi \equiv \theta$ and $\psi \equiv \neg\theta$.

We assume $|M| \geq 2$ for all models $\mathcal{M}$.

Proof.

$2 \implies 1$: By induction one shows $\chi, \neg\chi \models \bot$ for all χ.

Burgess' result: Let ϕ, ψ be sentences of $\mathcal{D}$. The following are equivalent:

1. ϕ and ψ are contradictory in that $\phi, \psi \models \bot$ (i.e. $\mathcal{M} \models \phi$ iff $\mathcal{M} \not\models \psi$).
2. There is a sentence $\theta \in \mathcal{D}$ such that $\phi \equiv \theta$ and $\psi \equiv \neg\theta$.

We assume $|M| \geq 2$ for all models $\mathcal{M}$.

Proof.

$2 \implies 1$: By induction one shows $\chi, \neg\chi \models \bot$ for all χ.

$1 \implies 2$: Let $\theta_0 := \forall x =(x)$. Given our assumption, $\theta_0 \equiv \bot$ and $\neg\theta_0 \equiv \bot$.
Let $\phi_0 := \phi \vee \theta_0$ and $\psi_0 := \psi \vee \theta_0$. Then:

$$\begin{array}{lllllllll} \phi_0 & \equiv & \phi \vee \theta_0 & \equiv & \phi \vee \bot & \equiv & \phi & & \\ \neg\phi_0 & \equiv & \neg(\phi \vee \theta_0) & \equiv & \neg\phi \wedge \neg\theta_0 & \equiv & \neg\phi \wedge \bot & \equiv & \bot \end{array}$$

Similarly $\psi_0 \equiv \psi$ and $\neg\psi_0 \equiv \bot$.

Burgess' result: Let ϕ, ψ be sentences of $\mathcal{D}$. The following are equivalent:

1. ϕ and ψ are contradictory in that $\phi, \psi \models \bot$ (i.e. $\mathcal{M} \models \phi$ iff $\mathcal{M} \not\models \psi$).
2. There is a sentence $\theta \in \mathcal{D}$ such that $\phi \equiv \theta$ and $\psi \equiv \neg\theta$.

We assume $|M| \geq 2$ for all models $\mathcal{M}$.

Proof.

$2 \implies 1$: By induction one shows $\chi, \neg\chi \models \bot$ for all χ.

$1 \implies 2$: Let $\theta_0 := \forall x =\!(x)$. Given our assumption, $\theta_0 \equiv \bot$ and $\neg\theta_0 \equiv \bot$.
Let $\phi_0 := \phi \vee \theta_0$ and $\psi_0 := \psi \vee \theta_0$. Then:

$$\begin{array}{ccccccccc} \phi_0 & \equiv & \phi \vee \theta_0 & \equiv & \phi \vee \bot & \equiv & \phi & & \\ \neg\phi_0 & \equiv & \neg(\phi \vee \theta_0) & \equiv & \neg\phi \wedge \neg\theta_0 & \equiv & \neg\phi \wedge \bot & \equiv & \bot \end{array}$$

Similarly $\psi_0 \equiv \psi$ and $\neg\psi_0 \equiv \bot$. By the separation theorem let η be first-order such that $\phi_0 \models \eta$ and $\psi_0 \models \neg\eta$. Let $\theta := \phi_0 \wedge (\neg\psi_0 \vee \eta)$. Then:

$$\begin{array}{llllll} \theta & \equiv \phi_0 \wedge (\neg\psi_0 \vee \eta) & \equiv \phi_0 \wedge (\bot \vee \eta) & \equiv \phi_0 \wedge \eta & \equiv \phi_0 & \equiv \phi \\ \neg\theta & \equiv \neg(\phi_0 \wedge (\neg\psi_0 \vee \eta)) & \equiv \neg\phi_0 \vee \neg(\neg\psi_0 \vee \eta) & \equiv \bot \vee (\neg\neg\psi_0 \wedge \neg\eta) & \equiv \psi_0 \wedge \neg\eta & \equiv \psi_0 \quad \equiv \psi \quad \square \end{array}$$

Kontinen and Väänänen's result: Let ϕ, ψ be formulas of $\mathcal{D}$ with free variables $x_1, \ldots, x_n$. The following are equivalent:

1. ϕ and ψ are contradictory in that $\phi, \psi \models \bot$ (i.e. $\mathcal{M} \models_X \phi$ and $\mathcal{M} \models_X \psi$ implies $X = \varnothing$).
2. There is a formula $\theta \in \mathcal{D}$ free variables $x_1, \ldots, x_n$ such that $\phi \equiv \theta$ and $\psi \equiv \neg\theta$.

Syntax of Aloni's **Bilateral state-based modal logic BSML**

$$\phi \quad := \quad p \mid \neg\phi \mid \phi\wedge\psi \mid \phi\vee\psi \mid \Diamond\phi \mid \Box\phi \mid \mathrm{NE}$$

I.e. the syntax of classical modal logic together with the **non-emptiness atom** NE.

Syntax of Aloni's **Bilateral state-based modal logic BSML**

$$\phi \quad := \quad p \mid \neg\phi \mid \phi \wedge \psi \mid \phi \vee \psi \mid \Diamond\phi \mid \Box\phi \mid \mathrm{NE}$$

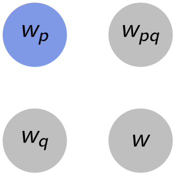
I.e. the syntax of classical modal logic together with the **non-emptiness atom** NE.

Modal team semantics: given a Kripke model $M = (W, R, V)$, a **team** of M is a set of possible worlds $s \subseteq W$:

standard Kripke semantics

$M, w \models \phi$

$w \in W$

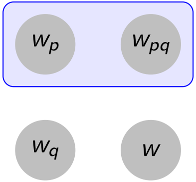


$w_p \models p$

team semantics

$M, s \models \phi$

$s \subseteq W$



$\{w_p, w_{pq}\} \models p$

Semantics:

$$
\begin{array}{lcl}
s \models p & \iff & \forall w \in s : w \in V(p) \\
s \dashv p & \iff & \forall w \in s : w \notin V(p) \\
\\
s \models \neg\phi & \iff & s \dashv \phi \\
s \dashv \neg\phi & \iff & s \models \phi \\
\\
s \models \phi \vee \psi & \iff & \exists t, t' : t \cup t' = s \text{ and } t \models \phi \text{ and } t' \models \psi \\
s \dashv \phi \vee \psi & \iff & s \dashv \phi \text{ and } s \dashv \psi \\
\\
s \models \Diamond\phi & \iff & \forall w \in s : \exists t \subseteq R[w] : t \neq \varnothing \text{ and } t \models \phi \\
s \dashv \Diamond\phi & \iff & \forall w \in s : R[w] \dashv \phi \\
\\
s \models \mathrm{NE} & \iff & s \neq \varnothing \\
s \dashv \mathrm{NE} & \iff & s = \varnothing
\end{array}
$$

where $R[w] = \{v \in W \mid wRv\}$. (We can define $\wedge := \neg \vee \neg$ and $\Box := \neg \Diamond \neg$.)

A team s represents the **information state** of a speaker.

A team s represents the **information state** of a speaker.

Bilateralism:

$s \models \phi$ represents **assertability** by a speaker in state s

$s \dashv \phi$ represents **rejectability** by a speaker in state s

BSML is designed to account for natural language phenomena such as **free choice inferences:**

You may have coffee or tea.

$\rightsquigarrow$You may have coffee and you may have tea.

Aloni (2022) conjectures that in certain situations speakers "systematically neglect structures which verify the sentence by virtue of some empty configuration." In **BSML** we can model this neglect of empty structures using NE. An account of free choice can then be made that relies on the fact that the following entailment holds: $\Diamond((c \wedge \mathrm{NE}) \vee (t \wedge \mathrm{NE})) \vDash \Diamond c \wedge \Diamond t$.

BSML is designed to account for natural language phenomena such as **free choice inferences:**

You may have coffee or tea.

$\rightsquigarrow$ You may have coffee and you may have tea.

Aloni (2022) conjectures that in certain situations speakers "systematically neglect structures which verify the sentence by virtue of some empty configuration." In **BSML** we can model this neglect of empty structures using NE. An account of free choice can then be made that relies on the fact that the following entailment holds: $\Diamond((c \wedge \mathrm{NE}) \vee (t \wedge \mathrm{NE})) \models \Diamond c \wedge \Diamond t$.

The bilateral negation is designed to ensure one gets correct predictions on natural language negation interacting with free choice inferences:

You may not have coffee or tea.

$\rightsquigarrow$ You may not have coffee and you may not have tea.

BSML$^{\mathbb{W}}$: **BSML** with the **global/inquisitive disjunction** $\mathbb{W}$:

$s \models \phi \mathbb{W} \psi$	iff	$s \models \phi$ or $s \models \psi$
$s \dashv \phi \mathbb{W} \psi$	iff	$s \dashv \phi$ and $s \dashv \psi$

BSML$^{\mathbb{W}}$: **BSML** with the **global/inquisitive disjunction** $\mathbb{W}$:

$s \models \phi \mathbin{\mathbb{W}} \psi$	iff	$s \models \phi$ or $s \models \psi$
$s \dashv \phi \mathbin{\mathbb{W}} \psi$	iff	$s \dashv \phi$ and $s \dashv \psi$

We also define the following abbreviations:

Weak contradiction $\bot := p \land \neg p$. $s \models \bot$ iff $s = \varnothing$.

Strong contradiction $\perp\!\!\!\perp := \bot \land \mathrm{NE}$. $s \models \perp\!\!\!\perp$ is never the case.

(Strong) tautology $\top := p \lor \neg p$. $s \models \top$ is always the case.

Some properties:

As with $\mathcal{D}$, we have failure of replacement for equivalents: $\bot \equiv \neg\mathrm{NE}$ but $p \vee \neg p \equiv \neg\bot \not\equiv \neg\neg\mathrm{NE} \equiv \mathrm{NE}$. Replacement succeeds for strong equivalents.

Some properties:

As with $\mathcal{D}$, we have failure of replacement for equivalents: $\bot \equiv \neg\mathrm{NE}$ but $p \vee \neg p \equiv \neg\bot \not\equiv \neg\neg\mathrm{NE} \equiv \mathrm{NE}$. Replacement succeeds for strong equivalents.

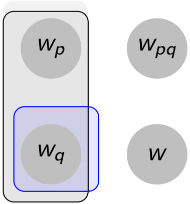
Formulas of classical modal logic ML (formulas without NE or $\mathbin{⩔}$) are flat: for $\alpha \in$ **ML**: $s \vDash \alpha$ iff $\forall w \in s : w \vDash \alpha$.

Some properties:

As with $\mathcal{D}$, we have failure of replacement for equivalents: $\bot \equiv \neg\mathrm{NE}$ but $p \vee \neg p \equiv \neg\bot \not\equiv \neg\neg\mathrm{NE} \equiv \mathrm{NE}$. Replacement succeeds for strong equivalents.

Formulas of classical modal logic ML (formulas without NE or $\mathbin{⩔}$) are flat: for $\alpha \in$ **ML**: $s \vDash \alpha$ iff $\forall w \in s : w \vDash \alpha$.

BSML is not downward closed and does not have the empty team property due to NE:



$$\begin{array}{lcl} \{w_p, w_q\} & \vDash & (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE}) \\ \{w_q\} & \nvDash & (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE}) \end{array}$$

The **bisimilarity** relation between pointed models captures equivalence with respect to **ML**.

(M, w) is a **Pointed model** (over a set of propositional symbols Φ) if M is a model over Φ and $w \in W$.

(M, w) and (M', w') (where both models are over supersets of Φ) being **k-bisimilar** (wrt Φ) $M, w \leftrightarrows_k^\Phi M', w'$ is defined recursively by:

$$w \leftrightarrows_0^\Phi w \iff \text{for all } p \in \Phi \text{ we have } w \models p \iff w' \models p.$$

$$w \leftrightarrows_{k+1}^\Phi w' \iff w \leftrightarrows_0^\Phi w' \text{ and}$$

[forth] for all $v \in R[w]$ there is a $v' \in R'[w']$ such that $v \leftrightarrows_{\Phi}{}_k v'$
[back] for all $v' \in R'[w']$ there is a $v \in R[w]$ such that $v \leftrightarrows_{\Phi}{}_k v'$

Modal depth of ϕ $(md(\phi))$: measure of the maximum nesting of $\Diamond$ in ϕ.

Let $P(\phi)$ be the set of proposition symbols used in ϕ.

(M, w) and (M', w') are **k-equivalent (wrt Φ)** $M, w \equiv_k^\Phi M', w'$ iff $w \models \phi \iff w' \models \phi$ for all ϕ with $md(\phi) \leq k$ and $P(\phi) \subseteq \Phi$

$$w \leftrightarrows_k^\Phi w' \iff w \equiv_k^\Phi w'$$

Hintikka formulas: characteristic formulas for worlds

$$\chi_{M,w}^{\Phi,0} \quad := \quad \bigwedge\{p \mid w \in V(p)\} \wedge \bigwedge\{\neg p \mid w \notin V(p)\} \qquad (p \in \Phi)$$

$$\chi_{M,w}^{\Phi,k+1} \quad := \quad \chi_{M,w}^{\Phi,k} \wedge \bigwedge_{v \in R[w]} \Diamond \chi_{M,v}^{\Phi,k} \wedge \Box \bigvee_{v \in R[w]} \chi_{M,v}^{\Phi,k}$$

$$w' \vDash \chi_{w}^{\Phi,k} \iff w \leftrightarrows_k^{\Phi} w' \iff w \equiv_k^{\Phi} w'$$

Hintikka formulas: characteristic formulas for worlds

$$\chi^{\Phi,0}_{M,w} := \bigwedge\{p \mid w \in V(p)\} \wedge \bigwedge\{\neg p \mid w \notin V(p)\} \quad (p \in \Phi)$$

$$\chi^{\Phi,k+1}_{M,w} := \chi^{\Phi,k}_{M,w} \wedge \bigwedge_{v \in R[w]} \Diamond \chi^{\Phi,k}_{M,v} \wedge \Box \bigvee_{v \in R[w]} \chi^{\Phi,k}_{M,v}$$

$$w' \vDash \chi^{\Phi,k}_{w} \iff w \leftrightharpoons^{\Phi}_{k} w' \iff w \equiv^{\Phi}_{k} w'$$

These can be used to define a disjunctive normal form for **ML**:

Property (over Φ): set of pointed models (over Φ).

Property (over Φ) defined by $\alpha \in$ ML: $|\alpha|_\Phi := \{(M, w) \text{ over } \Phi \mid w \vDash \alpha\}$.

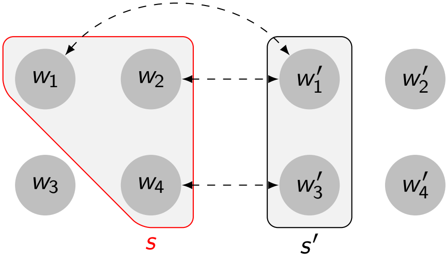
Normal form for **ML**: for $\alpha \in$ **ML**: for $\Phi \supseteq P(\alpha) : \alpha \equiv \bigvee_{(M,w) \in |\alpha|_\Phi} \chi^{\Phi,md(\alpha)}_{w}$.

Pointed (team) model (over Φ): (M, s) where s is a team on M, a model over Φ.

Pointed (team) model (over Φ): (M, s) where s is a team on M, a model over Φ.

Team bisimulation:

$s \rightleftharpoons_k^{\Phi} s' :\Longleftrightarrow$
forth: $\forall w \in s : \exists w' \in s' : w \rightleftharpoons_k^{\Phi} w'$



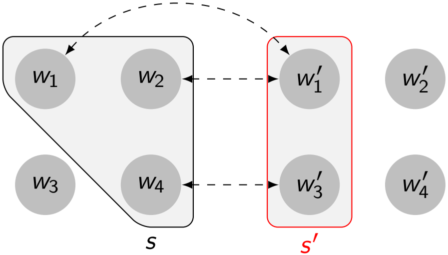
Pointed (team) model (over Φ): (M, s) where s is a team on M, a model over Φ.

Team bisimulation:

$s \rightleftharpoons_k^{\Phi} s' :\Longleftrightarrow$

forth: $\forall w \in s : \exists w' \in s' : w \rightleftharpoons_k^{\Phi} w'$

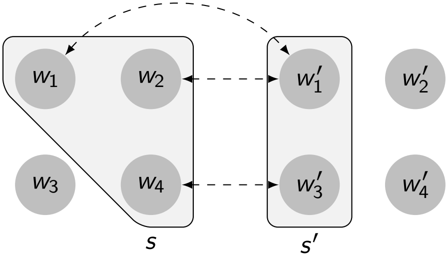
back: $\forall w' \in s' : \exists w \in s : w \rightleftharpoons_k^{\Phi} w'$



Pointed (team) model (over Φ): (M, s) where s is a team on M, a model over Φ.

Team bisimulation:

$$s \leftrightharpoons_k^{\Phi} s' :\Longleftrightarrow$$
$$\text{forth: } \forall w \in s : \exists w' \in s' : w \leftrightharpoons_k^{\Phi} w'$$
$$\text{back: } \forall w' \in s' : \exists w \in s : w \leftrightharpoons_k^{\Phi} w'$$



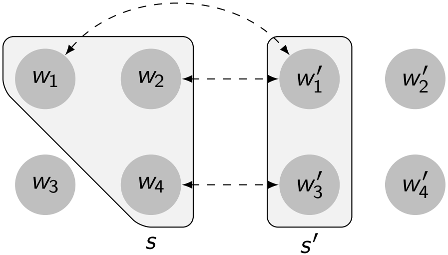
Characteristic formulas for teams:

$$\theta_{M,s}^{\Phi,k} := \bot \quad \text{if } s = \varnothing$$
$$\theta_{M,s}^{\Phi,k} := \bigvee_{w \in s} (\chi_{M,w}^{\Phi,k} \wedge \mathrm{NE}) \quad \text{if } s \neq \varnothing$$

Pointed (team) model (over Φ): (M, s) where s is a team on M, a model over Φ.

Team bisimulation:

$$s \leftrightharpoons_k^\Phi s' :\iff$$
forth: $\forall w \in s : \exists w' \in s' : w \leftrightharpoons_k^\Phi w'$
back: $\forall w' \in s' : \exists w \in s : w \leftrightharpoons_k^\Phi w'$



Characteristic formulas for teams:

$$\theta_{M,s}^{\Phi,k} := \bot \quad \text{if } s = \emptyset$$
$$\theta_{M,s}^{\Phi,k} := \bigvee_{w \in s} (\chi_{M,w}^{\Phi,k} \wedge \mathrm{NE}) \quad \text{if } s \neq \emptyset$$

$$s' \models \theta_s^{\Phi,k} \iff s \leftrightharpoons_k^\Phi s' \iff s \equiv_k^\Phi s'$$

Team property (over Φ): set of pointed team models (over Φ)

Property (over Φ) defined by ϕ $||\phi||_\Phi := \{(M,s) \text{ over } \Phi \mid s \models \phi\}$

Normal form for $\mathbf{BSML}^{\mathbb{W}}$: for $\phi \in \mathbf{BSML}^{\mathbb{W}}$: for $\Phi \supseteq P(\phi) : \phi \equiv \bigvee\!\!\!\!\bigvee_{(M,s)\in||\phi||_\Phi} \theta_s^{\Phi, md(\phi)}$.

Propositional fragments:

Team over Φ: a subset of 2^{Φ}.

Team property (over Φ): a subset of $\wp(2^{\Phi})$.

Property (over Φ) defined by ϕ $||\phi||_{\Phi} := \{s \subseteq 2^{\Phi} \mid s \models \phi\}$

Propositional characteristic formulas: let $p^{w(p)} = p$ if $w \models p$ and $p^{w(p)} = \neg p$ if $w \models \neg p$.

$$\chi_w^{\Phi} := \bigwedge_{p \in \Phi} p^{w(p)} \qquad v \models \chi_w^{\Phi} \iff v = w$$

$$\theta_s^{\Phi} := \bigvee_{w \in s} (\chi_w^{\Phi} \land \mathrm{NE}) \qquad t \models \theta_s^{\Phi} \iff s = t \qquad \phi \equiv \mathop{\bigvee\!\!\!\!\bigvee}_{s \in ||\phi||_{\Phi}} \bigvee_{w \in s} (\chi_w^{\Phi} \land \mathrm{NE}) \quad (\Phi \supseteq P(\phi))$$

In $\mathcal{D}$, we used the following as our notion of contradictoriness for the negation theorem:

$$
\begin{aligned}
\phi \text{ and } \psi \text{ are contradictory}_1 : \quad & \phi, \psi \vDash \bot \\
\iff \quad & \mathcal{M} \vDash_X \phi \text{ and } \mathcal{M} \vDash_X \psi \text{ implies } X = \varnothing \\
\iff \quad & \text{if } \phi, \psi \text{ are sentences: } \mathcal{M} \vDash \phi \iff \mathcal{M} \nvDash \psi
\end{aligned}
$$

In $\mathcal{D}$, we used the following as our notion of contradictoriness for the negation theorem:

$$\begin{array}{lll} \phi \text{ and } \psi \text{ are contradictory}_1: & & \phi, \psi \models \bot \\ & \iff & \mathcal{M} \models_X \phi \text{ and } \mathcal{M} \models_X \psi \text{ implies } X = \varnothing \\ & \iff & \text{if } \phi, \psi \text{ are sentences: } \mathcal{M} \models \phi \iff \mathcal{M} \not\models \psi \end{array}$$

This is not appropriate in a setting with NE and $\varovee$. Take $\phi := \bot \varovee (p \wedge \mathrm{NE})$ and $\psi := \bot \varovee ((p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE}))$.

In $\mathcal{D}$, we used the following as our notion of contradictoriness for the negation theorem:

$$
\begin{array}{lll}
\phi \text{ and } \psi \text{ are contradictory}_1 : & & \phi, \psi \models \bot \\
& \iff & \mathcal{M} \models_X \phi \text{ and } \mathcal{M} \models_X \psi \text{ implies } X = \varnothing \\
& \iff & \text{if } \phi, \psi \text{ are sentences: } \mathcal{M} \models \phi \iff \mathcal{M} \not\models \psi
\end{array}
$$

This is not appropriate in a setting with NE and $\mathbin{\mathbb{W}}$. Take $\phi := \bot \mathbin{\mathbb{W}} (p \wedge \mathrm{NE})$ and $\psi := \bot \mathbin{\mathbb{W}} ((p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE}))$. Then $\phi, \psi \models \bot$ so the negation result would give us θ s.t. $\theta \equiv \phi$ and $\neg\theta \equiv \psi$.

In $\mathcal{D}$, we used the following as our notion of contradictoriness for the negation theorem:

$$\begin{array}{lll} \phi \text{ and } \psi \text{ are contradictory}_1 : & & \phi, \psi \models \bot \\ & \Longleftrightarrow & \mathcal{M} \models_X \phi \text{ and } \mathcal{M} \models_X \psi \text{ implies } X = \varnothing \\ & \Longleftrightarrow & \text{if } \phi, \psi \text{ are sentences: } \mathcal{M} \models \phi \iff \mathcal{M} \not\models \psi \end{array}$$

This is not appropriate in a setting with NE and $\mathbin{\mathsf{W}}$. Take $\phi := \bot \mathbin{\mathsf{W}} (p \wedge \mathrm{NE})$ and $\psi := \bot \mathbin{\mathsf{W}} ((p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE}))$. Then $\phi, \psi \models \bot$ so the negation result would give us θ s.t. $\theta \equiv \phi$ and $\neg\theta \equiv \psi$. One can show:

Lemma: For all η: if $M, s \models \eta$ and $M, t \models \neg\eta$, then $s \cap t = \varnothing$.

But we have $\{w_p\} \models \phi$ and $\{w_p, w_{\neg p}\} \models \psi$ so $\{w_p\} \models \theta$ and $\{w_p, w_{\neg p}\} \models \neg\theta$. Therefore $\{w_p\} \cap \{w_p, w_{\neg p}\} = \{w_p\} = \varnothing$, a contradiction.

ϕ and ψ are contradictory$_1$: $\mathcal{M} \vDash_X \phi$ and $\mathcal{M} \vDash_X \psi$ implies $X = \varnothing$

Instead we essentially use (the modal analogue of) the following notion:

ϕ and ψ are contradictory$_2$: $\mathcal{M} \vDash_X \phi$ and $\mathcal{M} \vDash_Y \psi$ implies $X \cap Y = \varnothing$

ϕ and ψ are contradictory$_1$: $\mathcal{M} \vDash_X \phi$ and $\mathcal{M} \vDash_X \psi$ implies $X = \varnothing$

Instead we essentially use (the modal analogue of) the following notion:

ϕ and ψ are contradictory$_2$: $\mathcal{M} \vDash_X \phi$ and $\mathcal{M} \vDash_Y \psi$ implies $X \cap Y = \varnothing$

These are equivalent in the downward-closed setting of dependence logic:

Contradictory$_2$ always implies contradictory$_1$:
Let ϕ, ψ be contradictory$_2$. If $\mathcal{M} \vDash_X \phi$ and $\mathcal{M} \vDash_X \psi$ then $X \cap X = X = \varnothing$.

Contradictory$_1$ implies contradictory$_2$ if ϕ, ψ are downward closed:
Let ϕ, ψ be contradictory$_1$. If $\mathcal{M} \vDash_X \phi$ and $\mathcal{M} \vDash_Y \psi$, by downward closure $\mathcal{M} \vDash_{X \cap Y} \phi$ and $\mathcal{M} \vDash_{X \cap Y} \psi$ so $X \cap Y = \varnothing$.

ϕ and ψ are contradictory$_1$: $\mathcal{M} \vDash_X \phi$ and $\mathcal{M} \vDash_X \psi$ implies $X = \varnothing$

Instead we essentially use (the modal analogue of) the following notion:

ϕ and ψ are contradictory$_2$: $\mathcal{M} \vDash_X \phi$ and $\mathcal{M} \vDash_Y \psi$ implies $X \cap Y = \varnothing$

These are equivalent in the downward-closed setting of dependence logic:

Contradictory$_2$ always implies contradictory$_1$:
Let ϕ, ψ be contradictory$_2$. If $\mathcal{M} \vDash_X \phi$ and $\mathcal{M} \vDash_X \psi$ then $X \cap X = X = \varnothing$.

Contradictory$_1$ implies contradictory$_2$ if ϕ, ψ are downward closed:
Let ϕ, ψ be contradictory$_1$. If $\mathcal{M} \vDash_X \phi$ and $\mathcal{M} \vDash_Y \psi$, by downward closure $\mathcal{M} \vDash_{X \cap Y} \phi$ and $\mathcal{M} \vDash_{X \cap Y} \psi$ so $X \cap Y = \varnothing$.

The equivalence does not hold in our setting: $\bot \mathbin{⩔} (p \wedge \mathrm{NE})$ and $\bot \mathbin{⩔} ((p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE}))$ are (the modal analogue of) contradictory$_1$ but not contradictory$_2$.

Define:

$$|\phi|_\Phi \quad := \quad \{(M,w) \text{ over } \Phi \mid \exists s : w \in s \text{ and } M,s \models \phi\}$$

$|\phi|_\Phi$ is Hodges' notion of the flattening of ϕ; or the informative content of ϕ in inquisitive semantics.

For $\alpha \in \mathbf{ML}$, $|\alpha|_\Phi$ above coincides with our previous definition $|\alpha|_\Phi = \{(M,w)$ over $\Phi \mid M,w \models \alpha\}$.

In the propositional setting, $|\phi|_\Phi = \bigcup \|\phi\|_\Phi$.

ϕ and ψ are **contradictory** : $\iff$ $|\phi|_\Phi \cap |\psi|_\Phi = \varnothing$ (where $\Phi = P(\phi) \cup P(\psi)$)
$M,s \models \phi$ and $M,t \models \psi$ implies $s \cap t = \varnothing$

Separation theorem: If

ϕ and ψ are **contradictory** : $\quad |\phi|_\Phi \cap |\psi|_\Phi = \varnothing$ (where $\Phi = P(\phi) \cup P(\psi)$)

$\iff \quad M, s \vDash \phi$ and $M, t \vDash \psi$ implies $s \cap t = \varnothing$

then there is a $\eta \in$ **ML** such that $\phi \vDash \eta$ and $\psi \vDash \neg\eta$ and $P(\eta) = P(\phi) \cap P(\psi)$.

Separation theorem: If

ϕ and ψ are **contradictory** : $\quad |\phi|_\Phi \cap |\psi|_\Phi = \varnothing$ (where $\Phi = P(\phi) \cup P(\psi)$)

$\iff \quad M, s \vDash \phi$ and $M, t \vDash \psi$ implies $s \cap t = \varnothing$

then there is a $\eta \in$ **ML** such that $\phi \vDash \eta$ and $\psi \vDash \neg\eta$ and $P(\eta) = P(\phi) \cap P(\psi)$.

Proof (for the propositional fragment).

Recall that $\phi \equiv \mathbb{W}_{s \in ||\phi||_{P(\phi)}} \bigvee_{w \in s} (\chi_w^{P(\phi)} \wedge \mathrm{NE})$ and similarly for ψ.

Separation theorem: If

ϕ and ψ are **contradictory** : $\quad |\phi|_\Phi \cap |\psi|_\Phi = \varnothing$ (where $\Phi = P(\phi) \cup P(\psi)$)

$\iff \quad M, s \vDash \phi$ and $M, t \vDash \psi$ implies $s \cap t = \varnothing$

then there is a $\eta \in \mathbf{ML}$ such that $\phi \vDash \eta$ and $\psi \vDash \neg\eta$ and $P(\eta) = P(\phi) \cap P(\psi)$.

Proof (for the propositional fragment).

Recall that $\phi \equiv \mathbb{W}_{s \in ||\phi||_{P(\phi)}} \bigvee_{w \in s} (\chi_w^{P(\phi)} \wedge \mathrm{NE})$ and similarly for ψ.

Let $\eta_1 := \bigvee_{s \in ||\phi||_{P(\phi)}} \bigvee_{w \in s} \chi_w^{P(\phi)}$ and $\eta_2 := \bigvee_{s \in ||\psi||_{P(\psi)}} \bigvee_{w \in s} \chi_w^{P(\psi)}$.

Separation theorem: If

ϕ and ψ are **contradictory** : $\quad |\phi|_\Phi \cap |\psi|_\Phi = \varnothing$ (where $\Phi = P(\phi) \cup P(\psi)$)

$\iff \quad M, s \vDash \phi$ and $M, t \vDash \psi$ implies $s \cap t = \varnothing$

then there is a $\eta \in$ **ML** such that $\phi \vDash \eta$ and $\psi \vDash \neg\eta$ and $P(\eta) = P(\phi) \cap P(\psi)$.

Proof (for the propositional fragment).

Recall that $\phi \equiv \mathbb{W}_{s \in ||\phi||_{P(\phi)}} \bigvee_{w \in s} (\chi_w^{P(\phi)} \wedge \mathrm{NE})$ and similarly for ψ.

Let $\eta_1 := \bigvee_{s \in ||\phi||_{P(\phi)}} \bigvee_{w \in s} \chi_w^{P(\phi)}$ and $\eta_2 := \bigvee_{s \in ||\psi||_{P(\psi)}} \bigvee_{w \in s} \chi_w^{P(\psi)}$. Then $\phi \vDash \eta_1$ and $|\eta_1|_{P(\phi)} = |\phi|_{P(\phi)}$ (in fact, since η_1 is flat, $|\eta_1|_{P(\phi)} = \wp(|\phi|_{P(\phi)})$). Similarly $\psi \vDash \eta_2$ and $|\eta_2|_{P(\psi)} = |\psi|_{P(\psi)}$.

Separation theorem: If

$$\phi \text{ and } \psi \text{ are \textbf{contradictory}}: \quad |\phi|_\Phi \cap |\psi|_\Phi = \varnothing \text{ (where } \Phi = P(\phi) \cup P(\psi))$$
$$\iff \quad M, s \vDash \phi \text{ and } M, t \vDash \psi \text{ implies } s \cap t = \varnothing$$

then there is a $\eta \in \mathbf{ML}$ such that $\phi \vDash \eta$ and $\psi \vDash \neg\eta$ and $P(\eta) = P(\phi) \cap P(\psi)$.

Proof (for the propositional fragment).

Recall that $\phi \equiv \mathbb{W}_{s \in ||\phi||_{P(\phi)}} \bigvee_{w \in s} (\chi_w^{P(\phi)} \wedge \mathrm{NE})$ and similarly for ψ.

Let $\eta_1 := \bigvee_{s \in ||\phi||_{P(\phi)}} \bigvee_{w \in s} \chi_w^{P(\phi)}$ and $\eta_2 := \bigvee_{s \in ||\psi||_{P(\psi)}} \bigvee_{w \in s} \chi_w^{P(\psi)}$. Then $\phi \vDash \eta_1$ and $|\eta_1|_{P(\phi)} = |\phi|_{P(\phi)}$ (in fact, since η_1 is flat, $|\eta_1|_{P(\phi)} = \wp(|\phi|_{P(\phi)})$). Similarly $\psi \vDash \eta_2$ and $|\eta_2|_{P(\psi)} = |\psi|_{P(\psi)}$.

Locality: values for $\Phi \backslash P(\phi)$ do not affect the evaluation of ϕ (i.e. for $w \in 2^\Phi$, $w \vDash \phi \iff w \upharpoonright_{P(\phi)} \vDash \phi$).

Therefore $|\eta_1|_\Phi = |\phi|_\Phi \iff |\eta_1|_{P(\phi)} = |\phi|_{P(\phi)}$ and so $|\eta_1|_\Phi = |\phi|_\Phi$. Similarly $|\eta_2|_\Phi = |\psi|_\Phi$.

Separation theorem: If

$$\phi \text{ and } \psi \text{ are } \textbf{contradictory}: \quad |\phi|_\Phi \cap |\psi|_\Phi = \varnothing \text{ (where } \Phi = P(\phi) \cup P(\psi))$$
$$\iff \quad M, s \vDash \phi \text{ and } M, t \vDash \psi \text{ implies } s \cap t = \varnothing$$

then there is a $\eta \in \mathbf{ML}$ such that $\phi \vDash \eta$ and $\psi \vDash \neg\eta$ and $P(\eta) = P(\phi) \cap P(\psi)$.

Proof (for the propositional fragment).

Recall that $\phi \equiv \mathbb{W}_{s \in ||\phi||_{P(\phi)}} \bigvee_{w \in s} (\chi_w^{P(\phi)} \wedge \mathrm{NE})$ and similarly for ψ.

Let $\eta_1 := \bigvee_{s \in ||\phi||_{P(\phi)}} \bigvee_{w \in s} \chi_w^{P(\phi)}$ and $\eta_2 := \bigvee_{s \in ||\psi||_{P(\psi)}} \bigvee_{w \in s} \chi_w^{P(\psi)}$.Then $\phi \vDash \eta_1$ and $|\eta_1|_{P(\phi)} = |\phi|_{P(\phi)}$ (in fact, since η_1 is flat, $|\eta_1|_{P(\phi)} = \wp(|\phi|_{P(\phi)})$). Similarly $\psi \vDash \eta_2$ and $|\eta_2|_{P(\psi)} = |\psi|_{P(\psi)}$.

Locality: values for $\Phi \backslash P(\phi)$ do not affect the evaluation of ϕ (i.e. for $w \in 2^\Phi$, $w \vDash \phi \iff w \upharpoonright_{P(\phi)} \vDash \phi$).

Therefore $|\eta_1|_\Phi = |\phi|_\Phi \iff |\eta_1|_{P(\phi)} = |\phi|_{P(\phi)}$ and so $|\eta_1|_\Phi = |\phi|_\Phi$. Similarly $|\eta_2|_\Phi = |\psi|_\Phi$.

We show $\eta_1 \vDash \neg\eta_2$ (in standard single-valuation semantics). Assume $w \vDash \eta_1$.

Separation theorem: If

ϕ and ψ are **contradictory** : $\quad |\phi|_\Phi \cap |\psi|_\Phi = \varnothing$ (where $\Phi = P(\phi) \cup P(\psi)$)

$\iff \quad M, s \vDash \phi$ and $M, t \vDash \psi$ implies $s \cap t = \varnothing$

then there is a $\eta \in \mathbf{ML}$ such that $\phi \vDash \eta$ and $\psi \vDash \neg\eta$ and $P(\eta) = P(\phi) \cap P(\psi)$.

Proof (for the propositional fragment).

Recall that $\phi \equiv \mathbb{W}_{s \in ||\phi||_{P(\phi)}} \bigvee_{w \in s} (\chi_w^{P(\phi)} \wedge \mathrm{NE})$ and similarly for ψ.

Let $\eta_1 := \bigvee_{s \in ||\phi||_{P(\phi)}} \bigvee_{w \in s} \chi_w^{P(\phi)}$ and $\eta_2 := \bigvee_{s \in ||\psi||_{P(\psi)}} \bigvee_{w \in s} \chi_w^{P(\psi)}$.Then $\phi \vDash \eta_1$ and $|\eta_1|_{P(\phi)} = |\phi|_{P(\phi)}$ (in fact, since η_1 is flat, $|\eta_1|_{P(\phi)} = \wp(|\phi|_{P(\phi)})$). Similarly $\psi \vDash \eta_2$ and $|\eta_2|_{P(\psi)} = |\psi|_{P(\psi)}$.

Locality: values for $\Phi \backslash P(\phi)$ do not affect the evaluation of ϕ (i.e. for $w \in 2^\Phi$, $w \vDash \phi \iff w \restriction_{P(\phi)} \vDash \phi$).

Therefore $|\eta_1|_\Phi = |\phi|_\Phi \iff |\eta_1|_{P(\phi)} = |\phi|_{P(\phi)}$ and so $|\eta_1|_\Phi = |\phi|_\Phi$. Similarly $|\eta_2|_\Phi = |\psi|_\Phi$.

We show $\eta_1 \vDash \neg\eta_2$ (in standard single-valuation semantics). Assume $w \vDash \eta_1$.Then $w \in |\eta_1| = |\phi|$. Therefore $w \notin |\psi| = |\eta_2|$ and so $w \nvDash \eta_2$, whence $w \vDash \neg\eta_2$.

Separation theorem: If

ϕ and ψ are **contradictory** : $\quad |\phi|_\Phi \cap |\psi|_\Phi = \varnothing$ (where $\Phi = P(\phi) \cup P(\psi)$)

$\iff \quad M, s \vDash \phi$ and $M, t \vDash \psi$ implies $s \cap t = \varnothing$

then there is a $\eta \in$ **ML** such that $\phi \vDash \eta$ and $\psi \vDash \neg\eta$ and $P(\eta) = P(\phi) \cap P(\psi)$.

Proof (for the propositional fragment).

Recall that $\phi \equiv \mathbb{W}_{s \in ||\phi||_{P(\phi)}} \bigvee_{w \in s} (\chi_w^{P(\phi)} \wedge \mathrm{NE})$ and similarly for ψ.

Let $\eta_1 := \bigvee_{s \in ||\phi||_{P(\phi)}} \bigvee_{w \in s} \chi_w^{P(\phi)}$ and $\eta_2 := \bigvee_{s \in ||\psi||_{P(\psi)}} \bigvee_{w \in s} \chi_w^{P(\psi)}$.Then $\phi \vDash \eta_1$ and $|\eta_1|_{P(\phi)} = |\phi|_{P(\phi)}$ (in fact, since η_1 is flat, $|\eta_1|_{P(\phi)} = \wp(|\phi|_{P(\phi)})$). Similarly $\psi \vDash \eta_2$ and $|\eta_2|_{P(\psi)} = |\psi|_{P(\psi)}$.

Locality: values for $\Phi \backslash P(\phi)$ do not affect the evaluation of ϕ (i.e. for $w \in 2^\Phi$, $w \vDash \phi \iff w \upharpoonright_{P(\phi)} \vDash \phi$).

Therefore $|\eta_1|_\Phi = |\phi|_\Phi \iff |\eta_1|_{P(\phi)} = |\phi|_{P(\phi)}$ and so $|\eta_1|_\Phi = |\phi|_\Phi$. Similarly $|\eta_2|_\Phi = |\psi|_\Phi$.

We show $\eta_1 \vDash \neg\eta_2$ (in standard single-valuation semantics). Assume $w \vDash \eta_1$.Then $w \in |\eta_1| = |\phi|$. Therefore $w \notin |\psi| = |\eta_2|$ and so $w \nvDash \eta_2$, whence $w \vDash \neg\eta_2$.

Let η be the (classical) interpolant of η_1 and $\neg\eta_2$. Then $P(\eta) = P(\eta_1) \cap P(\eta_2) = P(\phi) \cap P(\psi)$ and $\phi \vDash \eta_1 \vDash \eta$ and $\psi \vDash \eta_2 \vDash \neg\eta$. $\square$

Lemma 1: For all η: if $M, s \vDash \eta$ and $M, t \vDash \neg\eta$, then $s \cap t = \varnothing$.

Lemma 2: For any ϕ there is a ϕ' such that $\phi \equiv \phi'$ and $\neg\phi' \nvDash \mathrm{NE}$.

Lemma 1: For all η: if $M, s \models \eta$ and $M, t \models \neg\eta$, then $s \cap t = \varnothing$.

Lemma 2: For any ϕ there is a ϕ' such that $\phi \equiv \phi'$ and $\neg\phi' \not\models \mathrm{NE}$.

Theorem: For any $\phi, \psi \in$ **BSML** (**BSML**$^{\mathbb{W}}$) the following are equivalent:

1. ϕ and ψ are contradictory in that $|\phi|_\Phi \cap |\psi|_\Phi = \varnothing$ ($\Phi = P(\phi) \cup P(\psi)$).
2. There is a $\theta \in$ **BSML** (**BSML**$^{\mathbb{W}}$) such that $\phi \equiv \theta$ and $\psi \equiv \neg\theta$.

Proof.

$2 \implies 1$: If $M, s \models \phi$ and $M, t \models \psi$, then $M, s \models \theta$ and $M, t \models \neg\theta$ so $s \cap t = \varnothing$ by Lemma 1.

Lemma 1: For all η: if $M, s \vDash \eta$ and $M, t \vDash \neg\eta$, then $s \cap t = \varnothing$.

Lemma 2: For any ϕ there is a ϕ' such that $\phi \equiv \phi'$ and $\neg\phi' \nvDash \mathrm{NE}$.

Theorem: For any $\phi, \psi \in$ **BSML** (**BSML**$^{\mathbb{W}}$) the following are equivalent:

1. ϕ and ψ are contradictory in that $|\phi|_\Phi \cap |\psi|_\Phi = \varnothing$ ($\Phi = P(\phi) \cup P(\psi)$).
2. There is a $\theta \in$ **BSML** (**BSML**$^{\mathbb{W}}$) such that $\phi \equiv \theta$ and $\psi \equiv \neg\theta$.

Proof.

$2 \implies 1$: If $M, s \vDash \phi$ and $M, t \vDash \psi$, then $M, s \vDash \theta$ and $M, t \vDash \neg\theta$ so $s \cap t = \varnothing$ by Lemma 1.

$1 \implies 2$: Let $\theta_0 := \Diamond(\bot\!\!\!\bot \vee \neg \bot\!\!\!\bot)$. Then:

$$\begin{array}{llllll} \theta_0 & = \Diamond(\bot\!\!\!\bot \vee \neg \bot\!\!\!\bot) & \equiv \Diamond \bot\!\!\!\bot & \equiv \bot & & \\ \neg\theta_0 & = \neg \Diamond (\bot\!\!\!\bot \vee \neg \bot\!\!\!\bot) & \equiv \Box\neg(\bot\!\!\!\bot \vee \neg \bot\!\!\!\bot) & \equiv \Box(\neg \bot\!\!\!\bot \wedge \neg\neg \bot\!\!\!\bot) & \equiv \Box(\neg \bot\!\!\!\bot \wedge \bot\!\!\!\bot) & \equiv \Box \bot\!\!\!\bot \quad \equiv \bot \end{array}$$

Lemma 1: For all η: if $M, s \vDash \eta$ and $M, t \vDash \neg\eta$, then $s \cap t = \varnothing$.

Lemma 2: For any ϕ there is a ϕ' such that $\phi \equiv \phi'$ and $\neg\phi' \nvDash \mathrm{NE}$.

Theorem: For any $\phi, \psi \in$ **BSML** (**BSML**$^{\mathbb{W}}$) the following are equivalent:

1. ϕ and ψ are contradictory in that $|\phi|_\Phi \cap |\psi|_\Phi = \varnothing$ ($\Phi = P(\phi) \cup P(\psi)$).
2. There is a $\theta \in$ **BSML** (**BSML**$^{\mathbb{W}}$) such that $\phi \equiv \theta$ and $\psi \equiv \neg\theta$.

Proof.

$2 \implies 1$: If $M, s \vDash \phi$ and $M, t \vDash \psi$, then $M, s \vDash \theta$ and $M, t \vDash \neg\theta$ so $s \cap t = \varnothing$ by Lemma 1.

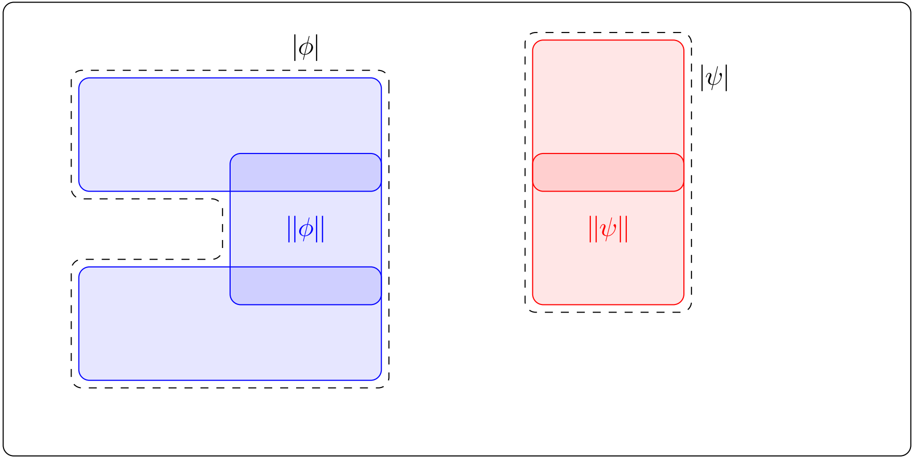
$1 \implies 2$: Let $\theta_0 := \Diamond(\perp\!\!\!\perp \vee \neg \perp\!\!\!\perp)$. Then:

$$\begin{array}{llllll} \theta_0 & = \Diamond(\perp\!\!\!\perp \vee \neg \perp\!\!\!\perp) & \equiv \Diamond \perp\!\!\!\perp & \equiv \bot & & \\ \neg\theta_0 & = \neg \Diamond (\perp\!\!\!\perp \vee \neg \perp\!\!\!\perp) & \equiv \Box\neg(\perp\!\!\!\perp \vee \neg \perp\!\!\!\perp) & \equiv \Box(\neg \perp\!\!\!\perp \wedge \neg\neg \perp\!\!\!\perp) & \equiv \Box(\neg \perp\!\!\!\perp \wedge \perp\!\!\!\perp) & \equiv \Box \perp\!\!\!\perp \quad \equiv \bot \end{array}$$

By the Lemma let ϕ', ψ' be such that $\phi' \equiv \phi$ and $\psi' \equiv \psi$ and $\neg\phi' \nvDash \mathrm{NE}$ and $\neg\psi' \nvDash \mathrm{NE}$. Let $\phi_0 := \phi' \vee \theta_0$ and $\psi_0 := \psi' \vee \theta_0$ so that $\phi_0 \equiv \phi' \equiv \phi$ and $\neg\phi_0 \equiv \neg\phi' \wedge \neg\theta_0 \equiv \bot$, and similarly for ψ_0. By the separation theorem let $\eta \in$ **ML** be such that $\phi_0 \vDash \eta$ and $\psi_0 \vDash \neg\eta$. Then letting $\theta := \phi_0 \wedge (\neg\psi_0 \vee \eta)$ we have $\theta \equiv \phi$ and $\neg\theta \equiv \psi$ as before. □

Theorem: For any $\phi, \psi \in$ **BSML** (**BSML**$^{\mathbb{W}}$) the following are equivalent:

1. ϕ and ψ are contradictory in that $|\phi|_\Phi \cap |\psi|_\Phi = \varnothing$ ($\Phi = P(\phi) \cup P(\psi)$).
2. There is a $\theta \in$ **BSML** (**BSML**$^{\mathbb{W}}$) such that $\phi \equiv \theta$ and $\psi \equiv \neg\theta$.



Alternatively, we can construct a formula for $\mathbf{BSML}^{\mathbb{W}}$ which does not use θ_0 (and does not require the use of modalities). Note that $\neg \perp\!\!\!\perp \equiv \neg\bot \vee \neg\mathrm{NE} \equiv \top$.

Let η be the separation formula and let $\theta' := \neg((\neg\phi \vee \perp\!\!\!\perp) \mathbin{\mathbb{W}} \neg((\neg\psi \vee \perp\!\!\!\perp) \mathbin{\mathbb{W}} \eta))$.

Alternatively, we can construct a formula for $\mathbf{BSML}^{\mathbb{W}}$ which does not use θ_0 (and does not require the use of modalities). Note that $\neg \perp\!\!\!\perp \equiv \neg\bot \vee \neg\mathrm{NE} \equiv \top$.

Let η be the separation formula and let $\theta' := \neg((\neg\phi \vee \perp\!\!\!\perp) \mathbin{\mathbb{W}} \neg((\neg\psi \vee \perp\!\!\!\perp) \mathbin{\mathbb{W}} \eta))$.We have:

$$\begin{array}{llllll}
\theta' & = \neg((\neg\phi \vee \perp\!\!\!\perp) \mathbin{\mathbb{W}} \neg((\neg\psi \vee \perp\!\!\!\perp) \mathbin{\mathbb{W}} \eta)) & \equiv \neg(\neg\phi \vee \perp\!\!\!\perp) \wedge ((\neg\psi \vee \perp\!\!\!\perp) \mathbin{\mathbb{W}} \eta) & \equiv (\phi \wedge \top) \wedge (\perp\!\!\!\perp \mathbin{\mathbb{W}} \eta) & \equiv \phi \wedge \eta & \equiv \phi \\
\neg\theta' & \equiv (\neg\phi \vee \perp\!\!\!\perp) \mathbin{\mathbb{W}} \neg((\neg\psi \vee \perp\!\!\!\perp) \mathbin{\mathbb{W}} \eta) & \equiv \perp\!\!\!\perp \mathbin{\mathbb{W}} (\neg(\neg\psi \vee \perp\!\!\!\perp) \wedge \neg\eta) & \equiv (\psi \wedge \top) \wedge \neg\eta & \equiv \psi \wedge \neg\eta & \equiv \psi
\end{array}$$

If the modalities, the quantifiers and $\mathbin{\rotatebox[origin=c]{180}{$\wedge$}\!\!\!\rotatebox[origin=c]{180}{$\wedge$}}$ are not used, this type of result is at least restricted somewhat; e.g.:

Propositional dependence logic $\mathcal{PD}$ with the dual negation:

$$\phi \quad := \quad p \mid \bot \mid \top \mid =(p_1, \ldots, p_n, p) \mid \neg\phi \mid \phi \wedge \psi \mid \phi \vee \psi$$

Define the **flattening** ϕ^f of $\phi \in \mathcal{PD}$ by $\phi^f = \phi[\top / =(p_1, \ldots, p_n, p)]$ (for all dependence atoms $=(p_1, \ldots, p_n, p)$). (Väänänen's syntactical notion of flattening.)

If the modalities, the quantifiers and $\mathbin{\rotatebox[origin=c]{90}{$\geqslant$}}$ are not used, this type of result is at least restricted somewhat; e.g.:

Propositional dependence logic $\mathcal{PD}$ with the dual negation:

$$\phi \quad := \quad p \mid \bot \mid \top \mid \, =(p_1, \ldots, p_n, p) \mid \neg\phi \mid \phi \wedge \psi \mid \phi \vee \psi$$

Define the **flattening** ϕ^f of $\phi \in \mathcal{PD}$ by $\phi^f = \phi[\top / =(p_1, \ldots, p_n, p)]$ (for all dependence atoms $=(p_1, \ldots, p_n, p)$). (Väänänen's syntactical notion of flattening.)

For a classical formula α: $|\alpha|_\Phi = \{v \in 2^\Phi \mid v(\alpha) = 1\}$ and $|\neg\alpha|_\Phi = 2^\Phi \backslash |\alpha|_\Phi$. So also $|\phi^f|_\Phi = 2^\Phi \backslash |\neg\phi^f|_\Phi$.

If the modalities, the quantifiers and $\mathbb{W}$ are not used, this type of result is at least restricted somewhat; e.g.:

Propositional dependence logic $\mathcal{PD}$ with the dual negation:

$$\phi \quad := \quad p \mid \bot \mid \top \mid\ =\!(p_1, \ldots, p_n, p) \mid \neg\phi \mid \phi \wedge \psi \mid \phi \vee \psi$$

Define the **flattening** ϕ^f of $\phi \in \mathcal{PD}$ by $\phi^f = \phi[\top/ =\!(p_1, \ldots, p_n, p)]$ (for all dependence atoms $=\!(p_1, \ldots, p_n, p)$). (Väänänen's syntactical notion of flattening.)

For a classical formula α: $|\alpha|_\Phi = \{v \in 2^\Phi \mid v(\alpha) = 1\}$ and $|\neg\alpha|_\Phi = 2^\Phi \backslash |\alpha|_\Phi$. So also $|\phi^f|_\Phi = 2^\Phi \backslash |\neg\phi^f|_\Phi$.

One can show that for all $\phi \in \mathcal{PD} : |\phi|_\Phi = |\phi^f|_\Phi$. (So Hodges' notion of flattening coincides with Väänänen's; this is not the case in FO/modal dependence logic.)

So in particular, if ϕ and ψ are such that $\theta \equiv \phi$ and $\neg\theta \equiv \psi$, then $|\phi|_\Phi = |\theta|_\Phi = |\theta^f|_\Phi = 2^\Phi \backslash |\neg\theta^f|_\Phi = 2^\Phi \backslash |(\neg\theta)^f|_\Phi = 2^\Phi \backslash |\neg\theta|_\Phi = 2^\Phi \backslash |\psi|_\Phi$.

Burgess' (2003) assessment of his theorem:

> *In recent years Hintikka and co-workers have revived a variant version of the logic of Henkin sentences under the label "independence-friendly" logic, have restated many theorems about existential second-order sentences for this "new" logic, and have made very large claims about the philosophical importance of the theorems thus restated. In discussion, pro and con, of such philosophical claims it has not been sufficiently emphasized that contrariety, the only kind of "negation" available, fails to correspond to any operation on classes of models. For this reason it seemed worthwhile to set down, in the form of the corollary above, a clear statement of just how total the failure is.*

Burgess' (2003) assessment of his theorem:

> *In recent years Hintikka and co-workers have revived a variant version of the logic of Henkin sentences under the label "independence-friendly" logic, have restated many theorems about existential second-order sentences for this "new" logic, and have made very large claims about the philosophical importance of the theorems thus restated. In discussion, pro and con, of such philosophical claims it has not been sufficiently emphasized that contrariety, the only kind of "negation" available, fails to correspond to any operation on classes of models. For this reason it seemed worthwhile to set down, in the form of the corollary above, a clear statement of just how total the failure is.*

Without weighing in on the philosophical debate, we briefly note that the above might be slightly misleading:

> All logical symbols corresponding to operations on classes of models in the way Burgess is after would seem to be tantamount to the semantics being compositional in a unilateral sense. But Hintikka (1996) repeatedly argued against compositionality.

Burgess' (2003) assessment of his theorem:

> *In recent years Hintikka and co-workers have revived a variant version of the logic of Henkin sentences under the label "independence-friendly" logic, have restated many theorems about existential second-order sentences for this "new" logic, and have made very large claims about the philosophical importance of the theorems thus restated. In discussion, pro and con, of such philosophical claims it has not been sufficiently emphasized that contrariety, the only kind of "negation" available, fails to correspond to any operation on classes of models. For this reason it seemed worthwhile to set down, in the form of the corollary above, a clear statement of just how total the failure is.*

Without weighing in on the philosophical debate, we briefly note that the above might be slightly misleading:

All logical symbols corresponding to operations on classes of models in the way Burgess is after would seem to be tantamount to the semantics being compositional in a unilateral sense. But Hintikka (1996) repeatedly argued against compositionality.

On the other hand, Hodges (1997) had already shown that IF logic has a compositional semantics. In this semantics, one takes the semantic value of a formula ϕ to be the pair $(||\phi||, ||\neg\phi||)$. Negation then corresponds to the operation of flipping the elements of the pair. If one accepts a bilateral/rejectionist view on negation, having the semantic value consist of both $||\phi||$ and $||\neg\phi||$ is as desired. Burgess then appears to take the correctness of the unilateral/assertionist view on negation for granted.

Burgess' (2003) assessment of his theorem:

> *In recent years Hintikka and co-workers have revived a variant version of the logic of Henkin sentences under the label "independence-friendly" logic, have restated many theorems about existential second-order sentences for this "new" logic, and have made very large claims about the philosophical importance of the theorems thus restated. In discussion, pro and con, of such philosophical claims it has not been sufficiently emphasized that contrariety, the only kind of "negation" available, fails to correspond to any operation on classes of models. For this reason it seemed worthwhile to set down, in the form of the corollary above, a clear statement of just how total the failure is.*

Without weighing in on the philosophical debate, we briefly note that the above might be slightly misleading:

All logical symbols corresponding to operations on classes of models in the way Burgess is after would seem to be tantamount to the semantics being compositional in a unilateral sense. But Hintikka (1996) repeatedly argued against compositionality.

On the other hand, Hodges (1997) had already shown that IF logic has a compositional semantics. In this semantics, one takes the semantic value of a formula ϕ to be the pair $(||\phi||, ||\neg\phi||)$. Negation then corresponds to the operation of flipping the elements of the pair. If one accepts a bilateral/rejectionist view on negation, having the semantic value consist of both $||\phi||$ and $||\neg\phi||$ is as desired. Burgess then appears to take the correctness of the unilateral/assertionist view on negation for granted.

Hintikka (1996) argued that "in any sufficiently rich language, there will be two different notions of negation present" — the dual negation $\neg$ and the contradictory negation $\sim$. He introduced a version of IF logic with $\sim$ (extended IF logic).

References

[1] Maria Aloni. Logic and conversation: the case of free choice. *Semantics and Pragmatics*, 15(5), 2022. doi: 10.3765/sp.15.5.

[2] Aleksi Anttila. The logic of free choice. axiomatizations of state-based modal logics. MSc thesis, University of Amsterdam, 2021.

[3] John P. Burgess. A remark on Henkin sentences and their contraries. *Notre Dame Journal of Formal Logic*, 44(3):185–188, 2003. doi: 10.1305/ndjfl/1091030856.

[4] Lauri Hella, Kerkko Luosto, Katsuhiko Sano, and Jonni Virtema. The expressive power of modal dependence logic. In Barteld Kooi Rajeev Goré and Agi Kurucz, editors, *Advances in Modal Logic*, volume 10, pages 294–312. College Publications, 2014. doi: 10.48550/arXiv.1406.6266.

[5] Jaakko Hintikka. *The Principles of Mathematics Revisited*. Cambridge University Press, 1996. doi: 10.1017/CBO9780511624919.

[6] Wilfrid Hodges. Compositional semantics for a language of imperfect information. *Logic Journal of the IGPL*, 5(4):539–563, 1997. doi: 10.1093/jigpal/5.4.539.

[7] Juha Kontinen and Jouko Väänänen. A remark on negation in dependence logic. *Notre Dame Journal of Formal Logic*, 52(1):55–65, 2011. doi: 10.1215/00294527-2010-036.

[8] Jouko Väänänen. *Dependence Logic: a New Approach to Independence Friendly Logic*. Cambridge University Press, 2007. doi: 10.1017/CBO9780511611193.